

CRITICAL HEEGAARD SURFACES OBTAINED BY SELF-AMALGAMATION

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ABSTRACT. Critical surfaces can be regarded as topological index 2 minimal surfaces which was introduced by David Bachman. In this paper we give a sufficiently condition and a necessary condition for self-amalgamated Heegaard surfaces to be critical.

1. INTRODUCTION

Let F be a properly embedded, separating surface with no torus components in a compact, orientable, irreducible 3-manifold M , dividing M into two submanifolds. Then the *disk complex*, $\Gamma(F)$, is defined as follows:

- (1) Vertices of $\Gamma(F)$ are isotopy classes of compressing disks for F .
- (2) A set of $m+1$ vertices forms an m -simplex if there are representatives for each that are pairwise disjoint.

David Bachman explored the information which is contained in the topology of $\Gamma(F)$ by defining the *topological index* of F [3]. If $\Gamma(F)$ is non-empty then the topological index of F is the smallest n such that $\pi_{n-1}(\Gamma(F))$ is non-trivial. If $\Gamma(F)$ is empty then F will have topological index 0. If F has a well-defined topological index (i.e. $\Gamma(F) = \emptyset$ or non-contractible) then we will say that F is a *topologically minimal surface*.

By definition, F has topological index 0 if and only if it is incompressible, and has topological index 1 if and only if it is strongly irreducible. Critical surfaces, which are also defined by David Bachman[1][4], can be regarded as topological index 2 minimal surfaces[4].

Definition 1.1. [4] F is *critical* if the compressing disks for F can be partitioned into two sets C_0 and C_1 , such that

- (1) for each $i = 0, 1$, there is at least one pair of disks $V_i, W_i \in C_i$ on opposite sides of F such that $V_i \cap W_i = \emptyset$;
- (2) if $V \in C_0$ and $W \in C_1$ are on opposite sides of F then $V \cap W \neq \emptyset$.

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Some critical Heegaard surfaces have been constructed by Jung Hoon Lee.

Theorem 1.2. [9] *The standard minimal genus Heegaard splitting of (closed orientable surface) $\times S^1$ is a critical Heegaard splitting.*

Jung Hoon Lee also showed that some critical Heegaard surfaces can be obtained by amalgamating two strongly irreducible Heegaard splittings.

Theorem 1.3. [9] *Let $X \cup_S Y$ be an amalgamation of two strongly irreducible Heegaard splittings $V_1 \cup_{S_1} W_1$ and $V_2 \cup_{S_2} W_2$ along homeomorphic boundary components of $\partial_- V_1$ and $\partial_- V_2$. Assume that V_2 is constructed from $\partial_- V_2 \times I$ by attaching only one 1-handle. If there exist essential disks $D_1 \subset W_1$ and $D_2 \subset W_2$ which persist into disjoint essential disks in Y and X respectively, then S is critical.*

The following theorem is the main result of this paper, which states that some critical Heegaard surfaces can be obtained by self-amalgamating strongly irreducible Heegaard splittings. Terms in the theorems will be defined in Section 2.

Theorem 1.4. *Suppose M is an irreducible 3-manifold with two homeomorphic boundary components F_1 and F_2 , and $V \cup_S W$ is a strongly irreducible Heegaard splitting of M such that $F_1 \cup F_2 \subset \partial_- W$. Suppose M admits an essential disk B in V and two spanning annuli A_1, A_2 in W , such that $\partial B, \partial_1 A_1, \partial_1 A_2$ are disjoint curves on S , and $\partial_2 A_i \subset F_i$, for $i = 1, 2$. Let $M^* = V^* \cup_{S^*} W^*$ be the self-amalgamation of $M = V \cup_S W$, such that $\partial_2 A_1$ is identified with $\partial_2 A_2$. Then S^* is a critical Heegaard surface of M^* .*

As a corollary, we show a generalized result of Theorem 1.2.

Corollary 1.5. *Let F be a closed, connected, orientable surface, and let $\varphi : F \rightarrow F$ be a surface diffeomorphism which preserves orientation. If $d(\varphi) \leq 2$, then the standard Heegaard surface of the surface bundle $M(F, \varphi)$ is critical.*

We also show a necessary condition for self-amalgamated Heegaard surfaces to be critical.

Theorem 1.6. *Suppose that $M^* = V^* \cup_{S^*} W^*$ is the self-amalgamation of $M = V \cup_S W$. If S^* is a critical Heegaard surface of M^* , then $d(S) \leq 2$.*

2. PRELIMINARIES

An essential annulus A properly embedded in a compression body C is called a *spanning annulus* if one component of ∂A denoted by $\partial_1 A$ lies in $\partial_+ C$, while the other denoted by $\partial_2 A$ lies in $\partial_- C$.

Let M be a compact orientable 3-manifold. If there is a closed surface S which cuts M into two compression bodies V and W with $S = \partial_+ V = \partial_+ W$, then we say M has a *Heegaard splitting*, denoted by $M = V \cup_S W$; and S is called a *Heegaard surface* of M .

A Heegaard splitting $M = V \cup_S W$ is said to be *reducible* if there are two essential disks $D_1 \subset V$ and $D_2 \subset W$ such that $\partial D_1 = \partial D_2$; otherwise, it is *irreducible*. A Heegaard splitting $M = V \cup_S W$ is said to be *weakly reducible* if there are two essential disks $D_1 \subset V$ and $D_2 \subset W$ such that $\partial D_1 \cap \partial D_2 = \emptyset$; otherwise, it is *strongly irreducible*.

The *distance* between two essential simple closed curves α and β in S , denoted by $d(\alpha, \beta)$, is the smallest integer $n \geq 0$ such that there is a sequence of essential simple closed curves $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$ in S such that α_{i-1} is disjoint from α_i for $1 \leq i \leq n$.

The *distance* of the Heegaard splitting $V \cup_S W$ is $d(S) = \text{Min}\{d(\alpha, \beta)\}$, where α bounds an essential disk in V and β bounds an essential disk in W . $d(S)$ was first defined by Hempel, see [7].

Let M be a compact orientable 3-manifold with homeomorphic boundary components F_1 and F_2 , and $M = V \cup_S W$ be a Heegaard splitting such that $F_1 \cup F_2 \subset \partial_- W$. Let M^* be the manifold obtained from M by gluing F_1 and F_2 via a homeomorphism $f : F_1 \rightarrow F_2$. Then M^* has a natural Heegaard splitting $M^* = V^* \cup_{S^*} W^*$ called the *self-amalgamation* of $M = V \cup_S W$ as follows:

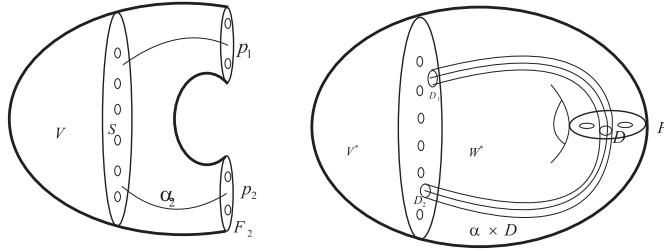
Let p_i be a point on F_i such that $f(p_1) = p_2$. Note that W is obtained by attaching 1-handles h_1, \dots, h_m to $\partial_- W \times I$. Let $\alpha_i = p_i \times I$, $\alpha_i \times D$ be the regular neighborhood of α_i for $i = 1, 2$. We may assume that $\alpha_i \times D$ is disjoint from the 1-handles h_1, \dots, h_m , and $f(p_1 \times D) = p_2 \times D$.

Now, in the closure of $M^* - V$, the arc $\alpha = \alpha_1 \cup \alpha_2$ has a regular neighborhood $\alpha \times D$ which intersects $\partial_+ V = S$ in two disks D_1 and D_2 . We denote by p the point p_i , D the disk $p \times D \subset \alpha \times D$, and F the surface F_i in M^* . Let $V^* = V \cup \alpha \times D$ and W^* be the closure of $M^* - V^*$. V^* and W^* are compression bodies. Let S^* be $V^* \cap W^*$, then $M^* = V^* \cup_{S^*} W^*$ is a Heegaard splitting called the self-amalgamation of $V \cup_S W$. It is clear that $g(S^*) = g(S) + 1$ (Fig.1).

Lemma 2.1. [11] $F - \text{int}D$ is incompressible in W^* .

Let S_1 be the surface $S - \text{int}D_1 \cup \text{int}D_2$. Then S_1 is a sub-surface of S with two boundary components ∂D_1 and ∂D_2 . An essential arc γ in S_1 is called *strongly essential* if both two boundary points lie in ∂D_i and γ is an essential arc on $S_1 \cup D_j$, where $\{i, j\} = \{1, 2\}$.

Lemma 2.2. [11] Suppose that E is an essential disk in V^* or W^* and $\partial E \cap \partial D \neq \emptyset$. Then there exist an arc $\gamma \in \partial E \cap S_1$ such that γ is strongly essential in S_1 .

FIGURE 1. $V \cup_S W$ and $V^* \cup_{S^*} W^*$

Lemma 2.3. [11] Suppose that E is an essential disk in V^* and $|E \cap D|$ is minimal up to isotopy. Let Δ be any outermost disk of E cut by $E \cap D$. Then $\partial\Delta \cap S_1$ is strongly essential in S_1 .

A *surface bundle*, denoted by $M(F, \varphi)$, is a 3-manifold obtained from $F \times [0, 1]$ by gluing its boundary components via a surface diffeomorphism $\varphi : F \times \{0\} \rightarrow F \times \{1\}$. When φ is the identity, $M(F, \varphi) \cong F \times S^1$.

Let F be a closed orientable surface with genus $g(F) \geq 2$. Suppose that φ is a homeomorphism of F . The *translation distance* of φ is $d(\varphi) = \min\{d(\alpha, \varphi(\alpha))\}$, where α is an essential simple closed curve on F . $d(\varphi)$ was first defined by Bachman and Schleimer [5].

3. PROOFS OF THEOREM 1.4 AND COROLLARY 1.5

Now we give the proof of Theorem 1.4. It shows a sufficient condition for a self-amalgamated Heegaard surface to be critical.

Proof. (of Theorem 1.4.) Since $V \cup_S W$ is strongly irreducible, it follows from Casson and Gordon's theorem [6] that F is incompressible. Since $\partial_2 A_1 = \partial_2 A_2$, it follows that $A_1 \cup A_2$ is an essential annulus in $M^* - V$, denoted by A . Take a spanning arc α in A , and let $V^* = V \cup \alpha \times D$ and W^* be the closure of $M^* - V^*$. Then $M^* = V^* \cup_{S^*} W^*$ is obtained by self-amalgamation of $M = V \cup_S W$. Now we prove that S^* is a critical Heegaard surface of M^* .

Let D be a compressing disk of V^* corresponding to the 1-handle $\alpha \times D$. We give a partition of the compressing disks for S^* , $C_0 \cup C_1$, as follows: (For the sake of convenience, in the following statement, “a disk in $V^* \cap C_i$ ” means “a compressing disk in V^* which belongs to C_i ”.)

$V^* \cap C_0$ consists of compressing disks in V^* that could be isotoped into V but inessential in V ;

$W^* \cap C_0$ consists of compressing disks in W^* that are disjoint from D ;

$V^* \cap C_1$ consists of compressing disks in V^* that do not belong to $V^* \cap C_0$;

$W^* \cap C_1$ consists of compressing disks in W^* that are not disjoint from D .

Each compressing disk of S^* must be contained in C_0 or C_1 . Now we need to show $C_0 \cup C_1$ satisfies the definition of criticality.

Claim 1. C_0 contains a disjoint pair of disks on opposite sides of S^* .

Note that D belongs to C_0 . Since $\partial_- W$ has two components, there exists at least one essential disk in W disjoint from $\alpha \times D$. Hence there exists at least one essential disk in W^* which is disjoint from D . This means C_0 contains disjoint compressing disks for S^* on opposite sides.

Claim 2. C_1 contains a disjoint pair of disks on opposite sides of S^* .

Since α is contained in the annulus A , $cl(A - (\alpha \times D))$ is an essential disk in W^* intersecting D in at least two points, so it belongs to C_1 . The essential disk B in V persists as an essential disk in V^* and belongs to C_1 . By assumption, $cl(A - \alpha \times D)$ is disjoint with B . This means C_1 also contains disjoint compressing disks for S^* on opposite sides.

Claim 3. Any disk in $V^* \cap C_0$ intersects any disk in $W^* \cap C_1$.

Let E be any disk in W^* that intersect with D . Let D_s be any disk essential in V^* , but inessential in V . Recall $D_1 \cup D_2 = (\alpha \times D) \cap S$. If ∂D_s is isotopic to one of ∂D_1 and ∂D_2 , then $D_s \cong D$ and there is nothing to prove. So we suppose that ∂D_s bounds a pair of pants together with ∂D_1 and ∂D_2 . By Lemma 2.2, there is an arc $\gamma \in \partial E \cap S_1$ such that γ is strongly essential in S_1 . Note that a strongly essential arc in S_1 must intersect with ∂D_s . Hence $E \cap D_s \neq \emptyset$.

Claim 4. Any disk in $W^* \cap C_0$ intersects any disk in $V^* \cap C_1$.

Let E_0 be an essential disk in W^* that is disjoint from D . After isotopy, ∂E_0 can be made disjoint from $\alpha \times D$. By Lemma 2.1 E_0 and $F - int D$ can be made disjoint by a standard innermost disk argument. This means that E_0 can be regarded as an essential disk in W . Let D^1 be an essential disk in V^* that belongs to C_1 . For proving Claim 4, we need to show $D^1 \cap E_0 \neq \emptyset$.

Suppose to the contrary that $D^1 \cap E_0 = \emptyset$. We assume that D^1 is chosen so that the number of components of intersection $|D \cap D^1|$ is minimal up to isotopy of D^1 , satisfying $E_0 \cap D^1 = \emptyset$. First, we suppose $|D \cap D^1| = \emptyset$. Then D^1 can be regard as an essential disk in V . Then $D_1 \cap E_0 \neq \emptyset$ since $V \cup_S W$ is strongly irreducible, a contradiction. Hence $|D \cap D^1| \neq \emptyset$. By a standard innermost disk argument, we can assume $D \cap D^1$ consists of arc components. Let $\beta \subset S^*$ be an outermost arc component in D^1 and Δ be the corresponding outermost disk in D^1 . Since the disk D cut V^* into V , after a small isotopy Δ lies in V .

By Lemma 2.3, $\beta \cap S_1$ is strongly essential in S_1 , hence Δ is essential in V . Since $V \cup_S W$ is strongly irreducible, $\partial \Delta \cap \partial E_0 \neq \emptyset$. It is easy to see that $\partial E_0 \cap \partial \Delta = \partial E_0 \cap \beta \subset \partial E_0 \cap \partial D^1$. However, we have assumed $E_0 \cap D^1 = \emptyset$, a contradiction. Claim 4 follows.

Hence $C_0 \cup C_1$ satisfies the definition of criticality. This completes the proof of Theorem 1.4. \square

Proof. (of Corollary 1.5.) The standard Heegaard splitting of a surface bundle[5] is the self-amalgamation of the type 2 Heegaard splitting of $\{\text{closed surface}\} \times I$ [10]. The genus of the Heegaard surface is $2g(F) + 1$. If $d(\varphi) \leq 2$, it is easy to see $M(F, \varphi)$ satisfies the condition of Theorem 1.4. \square

Remark 3.1. There are surface bundles of arbitrarily high genus which have genus two Heegaard splittings[8]. If $M(F, \varphi)$ contains a strongly irreducible Heegaard surface H , then $d(\varphi) \leq -\chi(H)$ [5]. It follows that if $M(F, \varphi)$ contains an irreducible genus two Heegaard surface, then it also contains a critical Heegaard surface.

4. A NECESSARY CONDITION FOR SELF-AMALGAMATED HEEGAARD SURFACES TO BE CRITICAL

The following result could be found in the proof for the main theorem in[11]. Recall we suppose $M^* = V^* \cup_{S^*} W^*$ is the self-amalgamation of $M = V \cup_S W$ and D is a meridian disk of V^* corresponding to the 1-handle attached to V .

Lemma 4.1. [11] *If $d(S) \geq 3$, for each pair of disks $D^* \subset V^*$ and $E^* \subset W^*$ such that D^* is not isotopic to D and $\partial E^* \cap \partial D \neq \emptyset$, we have $|D^* \cap E^*| \geq 2$.*

Proof. (of Theorem 1.6.) Since S^* is critical, the compressing disks for S^* can be partitioned into two sets C_0 and C_1 satisfying the definition of criticality.

Assume that $D \subset V^* \cap C_0$. Each disk in $V^* \cap C_1$ is not isotopic to D and each disk in $W^* \cap C_1$ intersects with D . Since S^* is critical, there exists at least one disjoint pair of disks $D^* \subset V^* \cap C_1$ and $E^* \subset W^* \cap C_1$. By Lemma 4.1, we have $d(S) \leq 2$. \square

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